

Post-inflationary preheating with weak coupling

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Particle production in the background of an external classical oscillating field is a key process describing the stage of preheating after inflation. For sufficiently strong couplings between the inflaton and matter fields, this process is known to proceed non-perturbatively. Parametric resonance plays crucial role for bosonic fields in this case, and the evolution of the occupation numbers for fermions is non-perturbative as well. In the Minkowski space, parametric resonance for bosons and non-perturbative effects for fermions would still persist even in the case of weak coupling. In particular, the energy density of created bosons would grow exponentially with time. However, the situation is quite different in the expanding universe. We give a simple demonstration how the conditions of the expanding universe, specifically, redshift of the field modes, lead to the usual perturbative expressions for particle production by an oscillating inflaton in the case of weak couplings. The results that we obtain are relevant and fully applicable to the Starobinsky model of inflation.

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I. INTRODUCTION

Reheating is one of the most important epoch of the universe evolution, connecting the inflation stage with the subsequent hot Big-Bang phase. Almost all matter constituting the universe at the subsequent radiation-dominated stage was created during the reheating stage. In most models of inflation based on the scalar field (inflaton), the universe is usually preheated by particle creation in the background of an oscillating inflaton. The particles created are subsequently thermalized, and the universe becomes hot. Initially, this process was treated perturbatively, e.g., by using the Born approximation for the decay rates of the inflaton (see [1]). later, it was realized [2–4] that creation of bosons may also proceed non-perturbatively via the effect of parametric resonance, and that the creation of fermions is non-perturbative as well [5]. This is especially true for sufficiently strong couplings between the inflaton and other fields, in which case the resonance is broad in the frequency space [3].

When the coupling between the inflaton and other fields is sufficiently weak, then, in some typical cases, it is considered legitimate to return to the usual Born perturbation theory in calculating the particle production rates *in the expanding universe* (see, e.g., [4]). This may look somewhat puzzling if one takes into account that the effect of parametric resonance for bosons, as well as the non-perturbative evolution of the occupation numbers of fermions, would certainly occur in the background of a classical field oscillating *in the Minkowski space-time*, however small is the coupling. In this case, therefore, the Born formula would not be applicable. Why, then, does it work in the space-time of an expanding universe? We clarify this issue in the present paper. We give a simple demonstration as to how the conditions of the expanding universe, notably, the redshift of frequencies of the field modes, result in the usual perturbative expressions for particle production by an oscillating inflaton in the case of weak couplings.

In this paper, we consider a model of inflation based on a power-law potential for the inflaton field, and the inflaton field throughout this paper will be regarded as classical. In Sec. II we describe the particle production via parametric resonance in the Minkowski space for the case in which a scalar field ϕ of mass M is weakly interacting (interaction constant ζ is sufficiently small) with a light scalar field φ of mass $m_\varphi \ll M$. We then describe several important modifications in the case of expanding universe, where ϕ plays the role of the inflaton, and particle creation becomes an essential part of the preheating process. In some

cases (violation the adiabaticity condition), parametric resonance does not develop in the expanding universe, and one returns to the Born approximation for the total width Γ_φ of decay of the field ϕ into a pair of φ particles, which would not be valid in the Minkowski space. In Secs. III and IV, we derive the Born approximation in the case of expanding universe, for bosons and fermions, respectively. In Sec. V, we show that the results obtained are relevant and fully applicable to the Starobinsky model of inflation [6]. In Sec. VI, we study the issue of the inflaton self-production in this model. Our general conclusions are formulated in Sec. VII.

II. PRELIMINARIES

Consider a scalar field ϕ of mass M interacting with a light scalar field φ of mass $m_\varphi \ll M$ with the interaction Lagrangian density

$$\mathcal{L}_{\text{int}} = -\zeta\phi\varphi^2, \quad (1)$$

where ζ is a constant with dimension mass. As our initial conditions, the homogeneous field $\phi(t)$ is classically oscillating in the neighborhood of its minimum at $\phi = 0$ with amplitude ϕ_0 , while the field φ is in the vacuum state. In the Minkowski space, this situation would lead to particle production via parametric resonance. Specifically, for sufficiently small values of ζ , namely, for

$$\zeta\phi_0 \ll M^2, \quad (2)$$

the resonance will be most efficient in the first narrow resonance band centered at the frequency

$$\omega_{\text{res}} = \frac{M}{2} \quad (3)$$

(see [2–4]). Within the resonance band, the mean particle occupation numbers grow with time according to the law

$$N_k = \frac{1}{1 - \Delta^2/\zeta^2\phi_0^2} \sinh^2 \lambda t, \quad (4)$$

where

$$\lambda = \frac{1}{M} \sqrt{\zeta^2\phi_0^2 - \Delta^2}, \quad \Delta = \omega_k^2 - \omega_{\text{res}}^2, \quad (5)$$

and $\omega_k = \sqrt{m_\varphi^2 + k^2} \approx k$ is the frequency of the mode of the field φ . The width of the resonance band of frequencies ω_k is determined by the condition that the expression under the square root in (5) is nonnegative.

The total particle number, as well as the energy density of the φ -particles, in Minkowski space grows asymptotically exponentially with time, in contrast to the expectations based on the naïve perturbation theory, where it grows with time only linearly.

There are several important modifications in the case of expanding universe, where ϕ plays the role of the inflaton, and particle creation is an essential part of the preheating process [2–4]. Firstly, the amplitude of the oscillating inflaton gradually decreases with time as $\phi_0 \propto a^{-3/2}$, where a is the scale factor. Secondly, the frequency of the mode of the scalar field φ is redshifted:

$$\omega_k = \sqrt{m_\varphi^2 + \frac{k^2}{a^2}} \approx \frac{k}{a}, \quad (6)$$

where k now is the comoving wave number, and we took into account that the mode is close to the resonance, hence, its wave number is relativistic.

One should note the difference of the picture in the Minkowski space and in the expanding universe. In the Minkowski space, the frequency picture is static. The modes within the resonance band will stay there all the time, and the resonant enhancement of the created particle number will occur in such modes. Thus, the resulting occupation numbers will grow according to (4). In the expanding universe, however, the frequency picture is dynamically changing: the frequencies of the modes of the field to be created are evolving due to the redshift. Each particular mode spends a finite time in the resonance zone, so that only a limited gain of the occupation number is possible. It is not clear *a priori* whether the resonant growth will occur or not in the expanding universe.

The theory of parametric resonance is applicable if the evolution of the relevant quantities occurs adiabatically. Specifically, if

$$\left| \frac{\dot{\phi}_0}{\phi_0} \right| = \frac{3}{2}H \ll M, \quad (7)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter, and if

$$\left| \frac{\dot{\lambda}}{\lambda} \right| \ll \lambda, \quad (8)$$

then one can replace law (4) by an approximate expression [4]

$$N_k \simeq \sinh^2 \int \lambda dt, \quad (9)$$

as long as the mode with the comoving wave number k remains within the resonance band.

If the adiabaticity condition (8) does not hold, and the parametric resonance, therefore, does not develop, then one usually employs the Born approximation for the total width Γ_φ of decay of a ϕ particle into a pair of φ particles:

$$\Gamma_\varphi = \frac{\zeta^2}{8\pi M}. \quad (10)$$

However, if this naïve formula does not work in the Minkowski space (as argued above), one may wonder why it works in the case of expanding universe, with continuously redshifted particle momenta etc.

Similar issues can be raised about the production of fermionic particles. Although there is no parametric resonance in this case, still the picture of creation of particle pairs by an oscillating *classical* field is quite different from that based on the usual perturbation theory [5]. Nevertheless, in the case of expanding universe, one often uses the Born formula for the total width of decay of ϕ into a pair $\bar{\psi}, \psi$:

$$\Gamma_\psi = \frac{\Upsilon^2 M}{8\pi} \quad (11)$$

where Υ is the Yukawa coupling of the scalar field ϕ to the fermionic field ψ .

The widths (10) and (11) in the Born approximation in the background of an oscillating classical field in the Minkowski space are calculated, e.g., in [4].

The purpose of this paper is to clarify the formulated issues and to show how equations (10) and (11) arise in the case of expanding universe. We then show that these equations are fully applicable to the Starobinsky model of inflation [6].

III. BOSONS

A scalar field φ with mass m_φ interacting with the inflaton ϕ via coupling (1) obeys the equation of motion

$$\square\varphi + (m_\varphi^2 + 2\zeta\phi)\varphi = 0. \quad (12)$$

Turning to the mode components φ_k one gets the equation

$$\ddot{\varphi}_k + 3H\dot{\varphi}_k + \left(\frac{k^2}{a^2} + m_\varphi^2 + 2\zeta\phi\right)\varphi = 0 \quad (13)$$

In order to eliminate the friction term, it is convenient to rescale the field as $\chi_k = a^{3/2}\varphi_k$. For the mode χ_k with the comoving wave number k , at the preheating stage, we have the

equation (see, e.g., [4])

$$\ddot{\chi}_k + \Omega_k^2 \chi_k = 0, \quad (14)$$

where

$$\Omega_k^2(t) = \omega_k^2(t) + 2\zeta\phi(t) - \frac{9}{4}H^2 - \frac{3}{2}\dot{H}, \quad (15)$$

and ω_k is given by (6). At the preheating stage, the inflaton field evolves as

$$\phi(t) = \phi_0(t) \cos Mt, \quad (16)$$

where the amplitude $\phi_0(t) \propto a^{-3/2}(t)$ slowly decreases with time due to the universe expansion as a consequence of the adiabaticity condition (7).

The equation for the Hubble parameter in an inflaton-dominated universe is

$$H^2 = \frac{1}{3M_{\text{P}}^2} \rho_\phi, \quad \rho_\phi = \frac{1}{2} M^2 \phi_0^2, \quad (17)$$

where

$$M_{\text{P}} = (8\pi G)^{-1/2} = 2.4 \times 10^{18} \text{ GeV} \quad (18)$$

is the reduced Planck mass. Under conditions (7), (8), the boson particle production proceeds via the effect of parametric resonance, as described in the preceding section. In this case, the last two terms in (15) can be neglected.

In this paper, we are interested in the case where condition (8) is *violated*, so that parametric resonance does not have time to develop and plays no role. Using equation (5), we see that, in the model under consideration, violation of condition (8) in the center of the resonance band is equivalent to

$$|\zeta| \lesssim \sqrt{\frac{3}{8}} \frac{M^2}{M_{\text{P}}}. \quad (19)$$

Since the inflaton-field oscillations mainly occur in the regime $\phi_0 \ll M_{\text{P}}$, we see that condition (2) is, in fact, a consequence of this inequality.

The effect of non-stationarity of the external field ϕ and of the metric is that the quantity Ω_k in the equation of motion (14) is a function of time. In the case $\Omega_k = \text{const}$, the solution for φ_k would maintain its positive-frequency character, i.e., $\varphi_k \sim e^{i\Omega_k t}$ for all t . The time-dependence of Ω_k results in the mixing of frequencies, hence, in particle production of the field φ . Under condition (19), parametric resonance does not play any role, and the particle occupation numbers are small. Hence, in calculating them, one is justified to use perturbation theory.

The mixing of frequencies is considered in a standard way by looking for solutions of the field equation in the form [7]

$$\varphi_k(t) = \frac{1}{\sqrt{\Omega_k}} \left[\alpha_k(t) e^{i \int_{t_0}^t \Omega_k(t') dt'} + \beta_k(t) e^{-i \int_{t_0}^t \Omega_k(t') dt'} \right], \quad (20)$$

$$\dot{\varphi}_k(t) = i\sqrt{\Omega_k} \left[\alpha_k(t) e^{i \int_{t_0}^t \Omega_k(t') dt'} - \beta_k(t) e^{-i \int_{t_0}^t \Omega_k(t') dt'} \right], \quad (21)$$

where $\alpha_k(t)$ and $\beta_k(t)$ are the Bogolyubov coefficients satisfying the relation

$$|\alpha_k|^2 - |\beta_k|^2 = 1. \quad (22)$$

In terms of these coefficients, the average occupation numbers in the corresponding modes are given by $N_k = |\beta_k|^2$. Thus, to find the number of created particles, one needs to find the coefficient β_k . Substituting expressions (20), (21) into (14), one obtains the following system of equations for α_k and β_k :

$$\dot{\alpha}_k = \frac{\dot{\Omega}_k}{2\Omega_k} e^{-2i \int_{t_0}^t \Omega_k(t') dt'} \beta_k, \quad (23)$$

$$\dot{\beta}_k = \frac{\dot{\Omega}_k}{2\Omega_k} e^{2i \int_{t_0}^t \Omega_k(t') dt'} \alpha_k. \quad (24)$$

The initial conditions for these equations are $\alpha_k = 1$, $\beta_k = 0$. Then, treating this system perturbatively, in the first order, the equation for coefficient β_k is

$$\dot{\beta}_k(t) = \frac{\dot{\Omega}_k}{2\Omega_k} e^{2i \int_{t_0}^t \Omega_k(t') dt'}. \quad (25)$$

Using equation (15) and employing the adiabaticity approximation (7), one transforms this equation into

$$\dot{\beta}_k = \left(\frac{\dot{\omega}_k}{2\omega_k} + \frac{\zeta M \phi_0}{2\omega_k^2} \sin Mt \right) e^{2i \int_{t_0}^t \omega_k(t') dt'}. \quad (26)$$

Expressing the function $\sin Mt$ as a sum of exponents, we get

$$\dot{\beta}_k = \frac{\dot{\omega}_k}{2\omega_k} e^{2i \int_{t_0}^t \omega_k(t') dt'} + \frac{\zeta M \phi_0}{4i\omega_k^2} e^{2i \int_{t_0}^t \omega_k(t') dt' + iMt} + \frac{\zeta M \phi_0}{4i\omega_k^2} e^{2i \int_{t_0}^t \omega_k(t') dt' - iMt}. \quad (27)$$

The first two terms oscillate with time with high frequency and thus do not give appreciable contribution to the coefficient β_k . The third term might pass through the resonance leading to the gain of β_k . Leaving only this resonant term, one obtains an approximate solution for the Bogolyubov coefficient β_k in the form

$$\beta_k = \frac{i\zeta M}{4} \int_{t_0}^t \frac{\phi_0(t')}{\omega_k^2(t')} e^{2i \int_{t_0}^{t'} \omega_k(t'') dt'' - iMt'} dt'. \quad (28)$$

As most particles are created in a narrow resonance region of frequencies, we can extend the limits of integration in (28) to infinity and use the stationary-phase approximation to estimate the value of integral. This gives

$$\beta_k = \frac{\zeta M \phi_0(t_k)}{4\omega_k^2(t_k)} \sqrt{\frac{\pi}{|\dot{\omega}_k(t_k)|}} = \frac{\zeta \phi_0(t_k)}{M} \sqrt{\frac{\pi}{|\dot{\omega}_k(t_k)|}} = \frac{\zeta \phi_0(t_k)}{M^{3/2}} \sqrt{\frac{2\pi}{H(t_k)}}, \quad (29)$$

where the moment of time t_k is defined by the stationary-phase relation $\omega_k(t_k) = M/2$, which is just the moment of passing through the center of the resonance band for the k -mode.

One can picture the process of particle creation in the following way. A mode with sufficiently high wave number k undergoes redshift till it reaches the resonance region. After passing through the narrow resonance band, it becomes filled with particles with average occupation numbers (29), which, in our approximation, remains subsequently constant. In this picture, the instantaneous particle spectrum is given by the following approximation:

$$N_k = \begin{cases} 0, & k > \frac{Ma(t)}{2} \quad (\text{the } k\text{-mode has not yet passed through resonance}), \\ |\beta_k|^2, & k_{\min} < k < \frac{Ma(t)}{2} \quad (\text{the } k\text{-mode has already passed through resonance}), \\ 0, & k < k_{\min} \quad (\text{the } k\text{-mode will never pass through resonance}). \end{cases} \quad (30)$$

All modes with momentum less than $k_{\min} = Ma(t_0)/2$, where t_0 is the moment of the beginning of particles creation, will never pass through the resonance region due to the redshift.

In this picture, the energy density $\rho_\varphi(t)$ of the created particles at any moment of time is given by

$$\rho_\varphi(t) = \frac{1}{a^4(t)} \int \frac{d^3k}{(2\pi)^3} \theta(k - k_{\min}) \theta(Ma(t) - 2k) k |\beta_k|^2, \quad (31)$$

where $\theta(x)$ is the Heaviside step function.

The effective rate of particle production Γ_φ is determined by comparing the time derivative of this energy density with the appropriate equation for the evolution of the energy density ρ_φ of continuously created relativistic particles

$$\dot{\rho}_\varphi = -4H\rho_\varphi + \Gamma_\varphi\rho_\varphi. \quad (32)$$

We have

$$\Gamma_\varphi\rho_\varphi = \left. \frac{k^3 |\beta_k|^2 M \dot{a}(t)}{4\pi^2 a^4(t)} \right|_{k=Ma(t)/2}, \quad (33)$$

whence, using (17) and (29), we get the standard expression (10) for the quantity Γ_φ .

IV. FERMIONS

In a curved space-time, one uses the covariant generalization of the Dirac equation:

$$[i\gamma^\mu(x)\mathcal{D}_\mu - m]\psi(x) = 0. \quad (34)$$

Here, $\gamma^\mu(x) = h_{(a)}^\mu(x)\gamma^a$, where γ^a are the usual Dirac matrices, and the tetrad vectors $h_{(a)}^\mu(x)$ are normalized as

$$h_{(a)}^\mu h_{(b)\mu} = \eta_{ab}, \quad (35)$$

with η_{ab} being the metric of the flat space.

The covariant derivative \mathcal{D}_μ of the Dirac field in (34) is defined as

$$\mathcal{D}_\mu\psi = \left[\partial_\mu + \frac{1}{4}C_{abc}h_\mu^{(c)}\gamma^b\gamma^a \right] \psi. \quad (36)$$

Here, C_{abc} are the Ricci coefficients, which are related to the tetrad vectors by

$$C_{abc} = \left(\partial_\nu h_{(a)}^\mu \right) h_{(b)\mu} h_{(c)}^\nu. \quad (37)$$

We consider the standard case where the spinor field ψ interacts with the inflaton field ϕ through the Yukawa coupling

$$\mathcal{L}_{\text{int}} = \Upsilon \phi \bar{\psi} \psi. \quad (38)$$

This results in the appearance of effective time-dependent fermion mass in equation (34):

$$m(t) = m_\psi - \Upsilon \phi(t). \quad (39)$$

It is convenient to proceed to the conformal time coordinate $\eta = \int a^{-1}(t)dt$, in terms of which the metric becomes conformally flat:

$$ds^2 = a^2(\eta) (d\eta^2 - d\vec{l}^2). \quad (40)$$

In cosmological setting, the tetrad vectors in the spherical coordinates can be chosen in the form

$$\begin{aligned} h_{(0)0} &= -h_{(1)1} = a(\eta), & h_{(2)2} &= -a(\eta)f(\chi), \\ h_{(3)3} &= -a(\eta)f(\chi)\sin\theta, & h_{(a)i} &= 0, \quad a \neq i. \end{aligned} \quad (41)$$

The Dirac equation (34) then reads [7]

$$\frac{i}{a} \left[\gamma^0 \frac{\partial}{\partial \eta} + \gamma^1 \frac{\partial}{\partial \chi} + \gamma^2 \frac{1}{f} \frac{\partial}{\partial \theta} + \gamma^3 \frac{1}{f \sin \theta} \frac{\partial}{\partial \phi} + \frac{3a'}{2a} \gamma^0 + \frac{f'}{f} \gamma^1 + \frac{\cot \theta}{2f} \gamma^2 \right] \psi - m\psi = 0, \quad (42)$$

where the prime denotes derivative with respect to the conformal time η .

Let us consider the case of pseudo-Euclidean space with $f = \chi$. In this case, it is convenient to use the Cartesian coordinates. Then equation (42) becomes

$$\left[\frac{i}{a} \gamma^\mu \partial_\mu + \frac{3i}{2a} \frac{a'}{a} \gamma^0 - m(\eta) \right] \psi = 0. \quad (43)$$

In order to separate variables in this equation, one can choose the solution in the form

$$\psi_j^{(\pm)} = \frac{1}{(2\pi a)^{3/2}} u_{k\sigma}^{(\pm)}(\eta) e^{\pm i k_\alpha x^\alpha} \quad (44)$$

Here, the collective quantum number $j = \{k, \sigma\}$, with $k = \{k_1, k_2, k_3\}$, $\sigma = \pm 1$, and

$$u_{\sigma k}^+ = \begin{pmatrix} \sigma f_{k+}(\eta) \varphi_\sigma(k) \\ -f_{k-}(\eta) \varphi_\sigma(k) \end{pmatrix}, \quad u_{\sigma k}^- = \begin{pmatrix} f_{k+}(\eta) \varphi_\sigma(k) \\ \sigma f_{k-}(\eta) \varphi_\sigma(k) \end{pmatrix}. \quad (45)$$

The spinors φ_σ are the eigenfunctions of the helicity operator

$$(\vec{\sigma} \vec{n}) \varphi_\sigma = \sigma \varphi_\sigma, \quad (46)$$

where $\vec{n} = \vec{k}/|\vec{k}|$, and $\vec{\sigma}$ are the Pauli matrices. Then, for the first component of equation (43), one has

$$\frac{i}{a} (\vec{\sigma} \vec{n}) \frac{ik}{(2\pi a)^{3/2}} f_{k-} \varphi_\sigma + \left(\frac{i}{a} \frac{\partial}{\partial \eta} + \frac{3i}{2a} \frac{a'}{a} - m \right) \left(\frac{\sigma}{(2\pi a)^{3/2}} f_{k+} \varphi_\sigma \right) = 0, \quad (47)$$

getting, for the time-dependent functions,

$$f'_{k+} + ik f_{k-} + ima f_{k+} = 0, \quad (48)$$

and

$$f'_{k-} + ik f_{k+} - ima f_{k-} = 0. \quad (49)$$

As in the scalar case, one can express the general solution of this system in terms of the Bogolyubov coefficients α_k and β_k as follows:

$$f_{k\pm}(\eta) = \pm N_{\mp}(\eta) \alpha_k(\eta) e^{i \int_{\eta_0}^{\eta} \Omega_k(\eta') a(\eta') d\eta'} - N_{\pm}(\eta) \beta_k(\eta) e^{-i \int_{\eta_0}^{\eta} \Omega_k(\eta') a(\eta') d\eta'}, \quad (50)$$

with

$$N_{\pm} = \sqrt{\frac{\Omega_k \pm m}{\Omega_k}}, \quad \Omega_k^2 = \frac{k^2}{a^2} + m^2. \quad (51)$$

The Bogolyubov coefficients satisfy the following system of equations:

$$\beta'_k = \frac{2ma' + m'a}{2k} e^{2i \int_{\eta_0}^{\eta} \Omega_k(\eta') a(\eta') d\eta'} \alpha_k, \quad (52)$$

$$\alpha'_k = \frac{2ma' - m'a}{2k} e^{-2i \int_{\eta_0}^{\eta} \Omega_k(\eta') a(\eta') d\eta'} \beta_k. \quad (53)$$

Proceeding to the usual cosmological time t , we can just repeat the derivation and the arguments of the preceding section with $\phi(t)$ in (39) given by (16). Eventually, we obtain the first-order perturbation-theory solution for the coefficient β_k in the form

$$\beta_k = \frac{\Upsilon M}{4i} \int_{t_0}^t \frac{\phi_0(t')}{\omega_k(t')} e^{2i \int_{t_0}^{t'} \omega_k(t'') dt'' - iMt'} dt', \quad (54)$$

where

$$\omega_k = \sqrt{m_\psi^2 + \frac{k^2}{a^2}} \approx \frac{k}{a}. \quad (55)$$

Higher-order corrections to this perturbative solution are small under the condition [4]

$$\frac{\Upsilon \phi_0}{M} \ll 1, \quad (56)$$

which is assumed to be the case.

By using the stationary-phase approximation, we obtain, similarly to (29),

$$\beta_k = -\frac{\Upsilon M \phi_0(t_k)}{4\omega(t_k)} \sqrt{\frac{\pi}{|\dot{\omega}_k(t_k)|}} = -\frac{1}{2} \Upsilon \phi_0(t_k) \sqrt{\frac{\pi}{|\dot{\omega}_k(t_k)|}} = -\frac{1}{2} \Upsilon \phi_0(t_k) \sqrt{\frac{2\pi}{MH(t_k)}}. \quad (57)$$

Then, repeating the reasoning of the end of Sec. III, and taking into account the four spin polarizations of particles and anti-particles, we get the final result for the production rate of fermions Γ_ψ , which coincides with (11).

V. THE STAROBINSKY MODEL

Historically, one of the first models that exhibited inflation was the model suggested by Starobinsky [6]. It is motivated by the necessity to consider local quantum corrections to the Einstein theory of gravity. The simplest such correction represents the term proportional to the second power of the Ricci scalar in the action of the model, so that the full gravitational action reads as

$$S_g = -\frac{M_P^2}{2} \int d^4x \sqrt{-g} \left(R - \frac{R^2}{6\mu^2} \right), \quad (58)$$

where

$$\mu = 1.3 \times 10^{-5} M_{\text{P}} \quad (59)$$

is a constant with indicated value required to explain the inflationary origin of the primordial perturbations [8].

The free scalar (φ) and spinor (ψ) fields are described by the usual actions

$$S_{\varphi} = \frac{1}{2} \int d^4x \sqrt{-g} (g^{\mu\nu} \partial_{\mu} \varphi \partial_{\nu} \varphi - m_{\varphi}^2 \varphi^2) , \quad (60)$$

$$S_{\psi} = \int d^4x \sqrt{-g} (i \bar{\psi} \mathcal{D} \psi - m_{\psi} \bar{\psi} \psi) . \quad (61)$$

A conformal transformation $g_{\mu\nu} \rightarrow \chi^{-1} g_{\mu\nu}$ with

$$\chi = \exp \left(\sqrt{\frac{2}{3}} \frac{\phi}{M_{\text{P}}} \right) \quad (62)$$

transforms the theory (58) into the usual Einstein gravity with a new special scalar field ϕ :

$$S_{\text{g}} = -\frac{M_{\text{P}}^2}{2} \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi - V(\phi) \right] , \quad (63)$$

where

$$V(\phi) = \frac{3\mu^2 M_{\text{P}}^2}{4} [1 - \chi^{-1}(\phi)]^2 \quad (64)$$

is the arising field potential.

The new scalar field ϕ (called the *scalaron*) universally interacts with other fields present in the theory. Thus, after the appropriate conformal transformation $\varphi \rightarrow \chi^{1/2} \varphi$, $\psi \rightarrow \chi^{3/4} \psi$, and $\mathcal{D} \rightarrow \chi^{1/2} \mathcal{D}$, the actions for scalar and fermion fields take the form, respectively,

$$S_{\varphi} = \int d^4x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_{\mu} \varphi \partial_{\nu} \varphi - \frac{1}{2\chi} m_{\varphi}^2 \varphi^2 + \frac{\varphi^2}{12M_{\text{P}}^2} g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi + \frac{\varphi}{\sqrt{6}M_{\text{P}}} g_{\mu\nu} \partial_{\mu} \varphi \partial_{\nu} \phi \right) , \quad (65)$$

$$S_{\psi} = \int d^4x \sqrt{-g} \left(i \bar{\psi} \mathcal{D} \psi - \frac{m_{\psi}}{\sqrt{\chi}} \bar{\psi} \psi \right) . \quad (66)$$

The inflationary model based on the scalaron ϕ is quite successful in solving the problems of the Big-Bang theory and is consistent with modern observations [9]. In this model, after the end of inflation, the scalaron starts oscillating near the minimum of its potential (64), which leads to production of particles in the external field of the oscillating scalaron. During most part of this stage, the condition

$$\left| \frac{\phi}{M_{\text{P}}} \right| \ll 1 \quad (67)$$

is valid, and, as the scalaron amplitude decreases, this inequality becomes stronger with time. Without taking into account the back-reaction of the matter fields on the dynamics of the scalaron field, its behavior at the stage of preheating is then approximately described by the Klein–Gordon equation

$$\square\phi + \mu^2\phi = 0, \quad (68)$$

and by the oscillatory regime (16) with mass $M = \mu$.

In this section, we are going to show that post-inflationary particle production in the model under consideration is well described by the theory developed in the preceding sections.

The equation of motion for the scalar φ field follows from (65):

$$\square\varphi + \left[\chi^{-1}(\phi) m_\varphi^2 + \frac{\square\phi}{\sqrt{6}M_{\text{P}}} - \frac{(\partial\phi)^2}{6M_{\text{P}}^2} \right] \varphi = 0, \quad (69)$$

or, using the equation of motion (68) of the scalaron in the neighborhood of its minimum,

$$\square\varphi + \left[\chi^{-1}(\phi) m_\varphi^2 - \frac{\mu^2\phi}{\sqrt{6}M_{\text{P}}} - \frac{(\partial\phi)^2}{6M_{\text{P}}^2} \right] \varphi = 0. \quad (70)$$

Due to condition (67), the last term in the brackets of (70) is much smaller than the previous term, so we can drop it. Assuming also that $m_\varphi \ll \mu$, we obtain an approximate equation

$$\square\varphi + \left[m_\varphi^2 - \frac{\mu^2\phi}{\sqrt{6}M_{\text{P}}} \right] \varphi = 0. \quad (71)$$

This is just equation (12) with

$$\zeta = -\frac{\mu^2}{2\sqrt{6}M_{\text{P}}}, \quad (72)$$

and one can check that condition (19) is satisfied.

Thus, the theory of Sec. III is applicable here, and the particle production rate is given by (10):

$$\Gamma_\varphi = \frac{\zeta^2}{8\pi M} = \frac{\mu^3}{192\pi M_{\text{P}}^2}, \quad (73)$$

which coincides with equation (7) of [9].

The spinor field ψ , under condition (67), interacts with the scalaron field ϕ through the Yukawa coupling (38) with strength

$$\Upsilon = \sqrt{\frac{2}{3}} \frac{m_\psi}{M_{\text{P}}}. \quad (74)$$

Again, one can see that the condition (56) for perturbative solution is satisfied due to the conditions $m_\psi \ll M = \mu$ and (67). We then calculate the relevant decay rate (11):

$$\Gamma_\psi = \frac{\Upsilon^2 M}{8\pi} = \frac{\mu m_\psi^2}{12\pi M_P^2}, \quad (75)$$

which, up to a factor four (possible account of the spin states), coincides with equation (8) of [9].

VI. INFLATON PRODUCTION

During preheating, inflaton field with nonlinear potential can produce its own excitations. To consider such self-production, we separate the inflaton field $\phi(\mathbf{x}, t)$ into the background part and excitation:

$$\phi(\mathbf{x}, t) = \phi(t) + \delta\phi(\mathbf{x}, t). \quad (76)$$

Substituting the field in this form into the equation of motion

$$\ddot{\phi} - \nabla^2 \phi + V'_\phi = 0, \quad (77)$$

one gets the equations for the background field and its excitation:

$$\ddot{\phi}(t) + V'_\phi(t) = 0, \quad (78)$$

$$\delta\ddot{\phi}_k(\mathbf{x}, t) + k^2 \delta\phi_k(\mathbf{x}, t) + V''_{\phi\phi} \delta\phi_k(\mathbf{x}, t) = 0. \quad (79)$$

Here, we have used the expansion

$$V'(\phi + \delta\phi) = V'(\phi) + V''(\phi)\delta\phi \quad (80)$$

and proceeded to the Fourier space.

At the reheating stage in the Starobinsky model, for $\phi/M_P \ll 1$, we have

$$V'' = \mu^2 \left(1 - \sqrt{6} \frac{\phi}{M_P} - \frac{4}{3} \left(\frac{\phi}{M_P} \right)^2 \right). \quad (81)$$

Then, restricting ourselves to the first order in ϕ/M_P , we obtain

$$\delta\ddot{\phi}_k + \left(\frac{k^2}{a^2} + \mu^2 - \sqrt{6}\mu^2 \frac{\phi}{M_P} \right) \delta\phi_k = 0. \quad (82)$$

This equation is similar to (14) with the background part of inflaton field behaving as in (16). Thus, we can treat (82) as described in Sec. III. For the Bogolyubov coefficient, we have the equation

$$\dot{\beta}_k = \frac{\dot{\Omega}_k}{2\Omega_k} e^{2i \int_{t_0}^t \Omega_k(t') dt'} \quad (83)$$

with

$$\Omega_k^2 = \omega_k^2(t) - \sqrt{6}\mu^2 \frac{\phi_0(t)}{M_P} \cos \mu t, \quad (84)$$

$$\omega_k^2 = \mu^2 + \frac{k^2}{a^2}. \quad (85)$$

In the approximation $\phi_0/M_P \ll 1$, we obtain

$$\frac{\dot{\Omega}_k}{2\Omega_k} = \frac{\sqrt{3}}{2\sqrt{2}} \frac{\mu^3 \phi_0}{\omega_k^2 M_P} \sin \mu t \quad (86)$$

and

$$\dot{\beta}_k = \frac{i\sqrt{3}}{4\sqrt{2}} \frac{\mu^3 \phi_0}{\omega_k^2 M_P} \left(e^{2i \int_{t_0}^t \Omega_k(t') dt' - i\mu t} - e^{2i \int_{t_0}^t \Omega_k(t') dt' + i\mu t} \right) \quad (87)$$

The specific feature of this equation is that the system does not pass through the resonance. Assuming that the pre-exponent coefficient in (87) changes slowly with time compared to the exponent, one can estimate the value of β_k , integrating only exponent, to obtain

$$|\beta_k|^2 \simeq \frac{3}{32} \frac{\phi_0^2}{M_P^2} \frac{\mu^4}{\omega_k^4}. \quad (88)$$

For the scalar particles, using (29) and (72), we have

$$|\beta_k|^2 = \frac{\pi \mu \phi_0^2}{12 M_P^2 H}. \quad (89)$$

Thus, the ratio of the production rates of the matter scalar particles and inflaton is approximately μ/H , and, since $H \ll \mu$, the production of inflaton is negligible. However, it might be comparable with the production of fermions of small mass with [see (57) and (74)]

$$|\beta_k|^2 = \frac{\pi}{3} \frac{m_\psi^2}{M_P^2} \frac{\phi_0^2}{\mu H}. \quad (90)$$

VII. DISCUSSION

Particle production in the background of an external classical oscillating field is one of the key processes describing the stage of preheating after inflation. Since the beginning of

1990s [2–4], it is known that this process often cannot be described by the usual decay rates of the inflaton calculated in perturbation theory. Thus, in Minkowski space, it would be dominated by the parametric resonance in the lowest resonance band no matter how small is the coupling between the inflaton and bosonic matter fields. The energy density of created particles would grow exponentially with time, in contrast to the usual perturbation-theory expectations. The process of creation of fermions would be described non-perturbatively as well [5].

The specific features of the expanding universe, surprisingly, restore the validity of the usual Born formula in the case of sufficiently small coupling. The reason is that every particular mode of the field to be excited spends only a finite amount of time in the resonance zone, resulting in small occupation numbers. Calculating the rates of particle production in the stationary-phase approximation, we obtain the standard classical formulas for the energy transfer between the inflaton and scalar and spinor fields in the case of small couplings. They are characterized by the standard effective decay rates (10) and (11), respectively, and do not depend on the details of the universe expansion. The reason of this peculiar property can be seen from equations (29) and (57). The mean occupation numbers of particles are $N_k = |\beta_k|^2 \propto 1/H(t_k)$, inversely proportional to the Hubble parameter at the time when the mode with wave number k passes through the resonance zone and gets excited. On the other hand, the rate of energy production is determined by the rate of filling new modes, which is governed by frequency redshift and is directly proportional to the Hubble parameter. These two effects compensate for each other leading to a constant effective values of Γ_φ and Γ_ψ .

It is also important that couplings (1) and (38) were linear in the inflaton field ϕ , so that we also had $N_k = |\beta_k|^2 \propto \phi_0^2 \propto \rho_\phi$. These are precisely the types of coupling responsible for the decay of a ϕ -particle, so that interpretation of (10) and (11) as decay rates makes sense.

We have shown that the considerations and results of the present paper are fully applicable to one of the most successful inflationary models — the Starobinsky model.

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